Morita Equivalence of Noncommutative Supertori

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ABSTRACT

In this paper we study the extension of Morita equivalence of noncommutative tori to the supersymmetric case. The structure of the symmetry group yielding Morita equivalence appears to be intact but its parameter field becomes supersymmetrized having both body and soul parts. Our result is mainly in the two dimensional case in which noncommutative supertori have been constructed recently: The group $SO(2, 2, V_{\mathbb{Z}}^0)$, where $V_{\mathbb{Z}}^0$ denotes Grassmann even number whose body part belongs to $\mathbb{Z}$, yields Morita equivalent noncommutative supertori in two dimensions.

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1   Introduction

In the work of Connes, Douglas, and Schwarz [1] Morita equivalence of two dimensional noncommutative tori resulting from toroidal compactifications of M(atrix) theory was mentioned in relation with T-duality in string theory [2, 3]. Then it was proved by Rieffel and Schwarz [4] that the actions of the group $SO(n,n,\mathbb{Z})$ on an antisymmetric $n \times n$ matrix θ which represents noncommutativity parameters for an n-dimensional noncommutative torus yield Morita equivalent tori. Then Schwarz [5] showed that compactifications on Morita equivalent tori are physically equivalent, corresponding to T-duality in string theory.

Noncommutative geometry [6] naturally appears in string theory: Low-energy effective theory of D-branes in a background NSNS B-field becomes the noncommutative field theory where the spacetime coordinates x^μ are noncommutative, $[x^\mu, x^\nu] \neq 0$ [1, 7, 8]. If we turn on the background RR field, the low-energy effective theory of D-branes becomes the field theory on non(anti)commutative superspace of which the fermionic coordinate θ^α has nontrivial commutation relation $\{\theta^\alpha, \theta^\beta\} \neq 0$ [9, 10, 11, 12, 13]. Gauge theories on non(anti)commutative superspace are studied extensively [14, 15, 16, 17]. Toroidal compactification in string theory with the above mentioned background field then naturally leads to noncommutative supertorus. Although the noncommutative torus has been a very well known subject for quite a long time, its supersymmetric version, the noncommutative supertorus, has been constructed only recently in the two dimensional case [18].

Recently, Berkovits and Maldacena [19] showed that tree level superstring theories on certain supersymmetric back grounds are related by the shift symmetry of a certain fermionic coordinate, which they called "fermionic" T-duality. This is very similar to bosonic T-duality in the sense that it is a symmetry under a shift of a coordinate. Bosonic T-duality is related with torus compactification. This naturally leads us to an expectation that the fermionic T-duality might be related to supertorus compactification. On the other hand, the fermionic T-duality relates different RR field backgrounds [19]. Furthermore, the background RR field leads to a non(anti)commutative superspace [12]. This makes us think that the fermionic T-duality might be related with the compactification on noncommutative supertorus rather than on commutative supertorus. In order to further investigate this idea, we have to understand Morita equivalence of noncommutative supertori first, since in the bosonic case it has been shown that T-duality corresponds to Morita equivalence of noncommutative tori [5]. This is the motivation behind our present work.

Commutative supertorus was constructed by Rabin and Freund [20] based on the work of Crane and Rabin [21] on super Riemann surfaces. The supertorus was obtained as a quotient of superplane by a subgroup of superconformal group Osp(1|2) which acts properly

discontinuously on the plane together with the metrizable condition. These two conditions boil down to proper latticing of the superplane, and can be expressed as appropriate translation properties along the cycles of the torus. Noncommutative tori can be constructed by embedding a lattice [22, 23, 24, 25] into Heisenberg group [26, 27, 28]. The lattice embedding determines how the generators of noncommutative torus, which correspond to the translation operators along the cycles of commutative torus, would act on the module of noncommutative torus. Heisenberg group can be regarded as a central extension of commutative space, which is equivalent to a deformation of space by constant noncommutativity. Super Heisenberg group can be regarded as a central extension of ordinary superspace, which is equivalent to a deformation of superspace by constant noncommutativity and nonanticommutativity.

Based on the construction of super Heisenberg group [29] as a central extension of ordinary superspace, the embedding maps for noncommutative supertori in two dimensions were obtained in [18]. Guided by the classical translation properties of commutative supertorus along its cycles, noncommutative supertorus was constructed in [18] by expressing the translations along the cycles of supertorus in operator language, and by implementing the spin structures of supertorus under even(bosonic) and odd(fermionic) translations with appropriate representations of spin angular momentum operator. Based on the above construction of noncommutative supertori, here we study Morita equivalence of noncommutative supertori by investigating symmetry actions that give dual embedding maps, i.e., finding the endomorphisms of the module that yield Morita equivalent tori.

This paper is organized as follows. In section 2, we review Morita equivalence in relation with the group $SO(n, n, \mathbb{Z})$ in the bosonic noncommutative n-tori case. In section 3, we recall the construction of noncommutative supertori. In section 4, we first consider the Morita equivalence of noncommutative supertori in a general setting and identify symmetry operators yielding Morita equivalent tori. When restricted to the two dimensional case, this yields the symmetry group $SO(2, 2, V_{\mathbb{Z}}^0)$ where $V_{\mathbb{Z}}^0$ denotes Grassmann even number whose body part belongs to $\mathbb{Z}$. We conclude in section 5.

2 Morita equivalence in the bosonic case

In this section we review Morita equivalence of noncommutative tori in the bosonic case in line with the work of Rieffel and Schwarz [4]. In general, Morita equivalence relates two algebras $\mathcal{A}$ and $\hat{\mathcal{A}}$ in such a way that for every $\mathcal{A}$-module R there exists $\hat{\mathcal{A}}$-module $\hat{R}$ where the correspondence $R \to \hat{R}$ is an equivalence of categories of $\mathcal{A}$-modules and $\hat{\mathcal{A}}$-modules. More specifically, (strong) Morita equivalence of C^*-algebra can be defined as follows. If we

consider a finite projective right module $\mathcal{E}$ over a C^*-algebra A, then the algebra $End_A\mathcal{E}$ of endomorphisms of $\mathcal{E}$ has a canonical structure as a C^*-algebra. We say that a C^*-algebra A' is (strongly) Morita equivalent to A if it is isomorphic to $End_A\mathcal{E}$ for some finite projective module $\mathcal{E}$.

An n-dimensional noncommutative torus is an associative algebra with involution having unitary generators $U_1, \ldots, U_n$ obeying the relations

$$U_i U_j = e^{2\pi i \theta_{ij}} U_j U_i, \quad i, j = 1, \ldots, n, \tag{1}$$

where (θ_{ij}) form a real $n \times n$ anti-symmetric matrix Θ. One can consider noncommutative torus as a C^*-algebra A_θ^n, the universal C^*-algebra generated by n unitary operators $\{U_j\}$ obeying the above relations. One can also consider a smooth version of A_θ^n consisting of formal series

$$\mathcal{A}_\theta^n = \{\sum a_{i_1 \cdots i_n} U_1^{i_1} \cdots U_n^{i_n} \mid a = (a_{i_1 \cdots i_n}) \in \mathcal{S}(\mathbb{Z}^n)\}, \tag{2}$$

where $\mathcal{S}(\mathbb{Z}^n)$ is Schwartz space of sequences with rapid decay.

Every projective module over a smooth algebra $\mathcal{A}_\theta^n$ can be represented by a direct sum of modules of the form $\mathcal{S}(\mathbb{R}^p \times \mathbb{Z}^q \times F)$ [22], the linear space of Schwartz functions on $\mathbb{R}^p \times \mathbb{Z}^q \times F$, where $2p + q = n$ and F is a finite abelian group. Let D be a lattice in $\mathcal{G} = M \times \hat{M}$, where $M = \mathbb{R}^p \times \mathbb{Z}^q \times F$ and $\hat{M}$ is its dual. The embedding map Φ under which D is the image of $\mathbb{Z}^n$ determines a projective module E on which the algebra of noncommutative torus acts.

In the Heisenberg representation the operators U's are defined by

$$U_{(m,\hat{s})} f(r) = e^{2\pi i <r,\hat{s}>} f(r + m), \quad m, r \in M, \ \hat{s} \in \hat{M}, \ f \in E, \tag{3}$$

where $< r, \hat{s} >$ is a usual inner product between M and $\hat{M}$. Here we would like to note that the vector $(m, \hat{s})$ can be mapped into an element of Heisenberg group which we explain below. In this representation, we can easily get the antisymmetric matrix Θ characterizing noncommutative torus as follows.

$$\begin{aligned} U_{(m,\hat{s})} U_{(n,\hat{t})} f(r) &= e^{2\pi i (<r,\hat{s}+\hat{t}>+<m,\hat{t}>)} f(r + m + n), \\ &= e^{2\pi i <m,\hat{t}>} U_{(m+n,\hat{s}+\hat{t}))} f(r), \end{aligned} \tag{4}$$

where $m, n, r \in M$, $\hat{s}, \hat{t} \in \hat{M}$, $f \in E$, and thus

$$U_{(m,\hat{s})} U_{(n,\hat{t})} = e^{2\pi i (<m,\hat{t}>-<n,\hat{s}>)} U_{(n,\hat{t})} U_{(m,\hat{s})}. \tag{5}$$

Therefore, if we denote $\vec{e}_i := (m, \hat{s})$, $\vec{e}_j := (n, \hat{t})$, then we can express θ_{ij} as follows.

$$\theta_{ij} = \vec{e}_i \cdot J_0 \vec{e}_j, \quad \text{where} \quad J_0 = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}. \tag{6}$$

Notice that with the embedding map Φ, which is given by $\Phi = (\vec{e}_1, \vec{e}_2, \cdots, \vec{e}_n)$, the above relation can be expressed in terms of Θ as

$$\Theta = \Phi^t J_0 \Phi, \quad \text{where} \quad J_0 = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}. \tag{7}$$

When M in (3) is given by $M = \mathbb{R}^p$, Heisenberg group $Heis(\mathbb{R}^{2p}, \psi)$ is defined as follows. For $t, t' \in U(1)$, and $(u, v), (u', v') \in \mathbb{R}^{2p}$, we define the product for $(t, u, v), (t', u', v') \in Heis(\mathbb{R}^{2p}, \psi)$,

$$(t, u, v) \cdot (t', u', v') = (t + t' + \psi(u, v; u', v'), u + u', v + v'), \tag{8}$$

where $\psi : \mathbb{R}^{2p} \times \mathbb{R}^{2p} \longrightarrow \mathbb{R}$, satisfies the cocycle condition

$$\psi(u, v; u', v')\psi(u + u', v + v'; u'', v'') = \psi(u, v; u' + u'', v' + v'')\psi(u', v'; u'', v''), \tag{9}$$

which is a necessary and sufficient condition for the multiplication to be associative. There is an exact sequence

$$0 \to \mathbb{R} \xrightarrow{i} Heis(\mathbb{R}^{2p}, \psi) \xrightarrow{j} \mathbb{R}^{2p} \to 0, \tag{10}$$

which is called a central extension, with the inclusion $i(t) = (t, 0, 0)$ and the projection $j(t, x, y) = (x, y)$, where $i(\mathbb{R})$ is the center in $Heis(\mathbb{R}^{2p}, \psi)$. The previously appeared vector $(m, \hat{s})$ in (3) corresponds to a vector $(u, v) \in \mathbb{R}^{2p}$ in the above description of Heisenberg group.

In a canonical form, we can define the embedding map as

$$\Phi = \begin{pmatrix} \widehat{\Theta} & 0 \\ 0 & I \end{pmatrix} := (x_{i,j}), \tag{11}$$

where $\widehat{\Theta} = \text{diag}(\theta_1, \cdots, \theta_p), \quad i, j = 1, \cdots, 2p$. The Heisenberg representation is given by

$$(U_j f)(s_1, \cdots, s_p) := (U_{\vec{e}_j} f)(\vec{s}) = \exp(2\pi i \sum_{k=1}^{p} s_k x_{k+p,j}) f(\vec{s} + \vec{x}_j), \quad \text{for} \quad j = 1, \cdots, 2p, \tag{12}$$

where $\vec{e}_j = (x_{1,j}, \ldots, x_{2p,j})$, $\vec{s} = (s_1, \cdots, s_p)$, and $\vec{x}_j = (x_{1,j}, \cdots, x_{p,j})$. Here, U_j's satisfy

$$U_j U_{j+p} = e^{2\pi i \theta_j} U_{j+p} U_j, \quad \text{otherwise} \quad U_j U_k = U_k U_j. \tag{13}$$

Applying the relation (7), one can also easily check the above relation:

$$\Theta = \begin{pmatrix} \widehat{\Theta} & 0 \\ 0 & I \end{pmatrix}^t \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \begin{pmatrix} \widehat{\Theta} & 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} 0 & \widehat{\Theta} \\ -\widehat{\Theta} & 0 \end{pmatrix}. \tag{14}$$

In the general embedding case, one can also use the Manin's representation [24] in which (12) becomes

$$(U_{\vec{e}_j} f)(\vec{s}) := \exp(2\pi i \sum_{k=1}^{p} s_k x_{k+p,j} + \frac{1}{2} \sum_{k=1}^{p} x_{k,j} x_{k+p,j}) f(\vec{s} + \vec{x}_j), \quad \text{for} \quad j = 1, \cdots, 2p. \quad (15)$$

This corresponds to the second representation of Heisenberg group in Ref. [18].

In order to consider the endomorphisms of A_θ^n, we first consider the group $O(n, n, \mathbb{R})$ on the space of $\mathcal{T}_n$ of real antisymmetric $n \times n$ matrices. The group $O(n, n, \mathbb{R})$ can be considered as a group of linear transformations of the space $\mathbb{R}^{2n}$ preserving the quadratic form $x_1 x_{n+1} + x_2 x_{n+2} + \cdots + x_n x_{2n}$. It is convenient to consider coordinates in $\mathbb{R}^{2n}$ as two n-dimensional vectors $(a^1, \ldots, a^n, b_1, \ldots, b_n)$ so that the quadratic form on $\mathbb{R}^{2n}$ can be written as $a^i b_i$. Thus we write the elements of $O(n, n, \mathbb{R})$ in 2×2 block form

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}. \quad (16)$$

The blocks A, B, C, D are $n \times n$ matrices satisfying $A^t C + C^t A = B^t D + D^t B = 0$, $A^t D + C^t B = 1$, where ^{t} denotes transpose. The action of $O(n, n, \mathbb{R})$ on the space $\mathcal{T}_n$ is defined by the formula

$$\Theta' = g\Theta := (A\Theta + B)(C\Theta + D)^{-1}. \quad (17)$$

Note that the above action is defined only on the subset of $\mathcal{T}_n$ where $C\Theta + D$ is invertible.

Now, let

$$\mathcal{T}_n^0 = \{\Theta \in \mathcal{T}_n : g\Theta \text{ is defined for all } g \in SO(n, n, \mathbb{Z})\}. \quad (18)$$

Then the following theorem holds [4].

Theorem. *For $\Theta \in \mathcal{T}_n^0$ and $g \in SO(n, n, \mathbb{Z})$ the noncommutative torus corresponding to $g\Theta$ is Morita equivalent to the noncommutative torus corresponding to Θ.*

This remains true for the smooth version of noncommutative tori.

Now we briefly review the proof of the above theorem. For the proof of the above theorem, we first consider a suitable set of generators for $SO(n, n, \mathbb{Z})$. In [4], it was shown as a lemma that the three elements, $\rho(R), \nu(N)$, and σ_2, which we explain below generate the group $SO(n, n, \mathbb{Z})$.

1) For every matrix $R \in GL(n, \mathbb{Z})$, $\rho(R) \in SO(n, n, \mathbb{Z})$ defines the following transformations

$$a'^i = R^i_j a^j, \quad b'_i = (R^{-1})^j_i b_j. \quad (19)$$

The action of $\rho(R)$ on $x = (a, b)$ with $a, b \in \mathbb{R}^n$ can be simply expressed as

$$\rho(R)x = \begin{pmatrix} R^t & 0 \\ 0 & R^{-1} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}. \quad (20)$$

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2) For an antisymmetric $n \times n$ matrix N whose entries are $n_{ij} \in \mathbb{Z}$ the transformation $\nu(N) \in SO(n, n, \mathbb{Z})$ is defined as follows.

$$a'^i = a^i + n^{ij}b_j, \quad b'_i = b_i. \tag{21}$$

3) For every integer k an element $\sigma_k \in O(n, n, \mathbb{Z})$ defines the following transformations

$$a'^i = b_i \ \text{ for } 1 \le i \le k, \quad a'^i = a^i \ \text{ for } k < i \le n,$$
$$b'_i = a^i \ \text{ for } 1 \le i \le k, \quad b'_i = b_i \ \text{ for } k < i \le n. \tag{22}$$

Note that $\sigma_k \in O(n, n, \mathbb{Z})$ only when k is even, since no element of $\mathcal{T}_n$ is invertible when n is odd. We refer the proof of the above lemma to Ref. [4], and proceed to the proof of the theorem. The proof of the theorem was done by showing that each action of the above generators for $SO(n, n, \mathbb{Z})$ yields Morita equivalent torus.

For $g = \rho(R)$, the noncommutative torus determined by $\Theta' = R\Theta R^t$ is Morita equivalent to the torus corresponding to Θ. This is because the two tori are isomorphic as we see below. For a general embedding, one can express an embedding vector $\vec{x}$ in terms of basis vectors $\{\vec{e_i}\}$ as $\vec{x} = \sum_1^n x_i \vec{e_i}$ where $x_i \in \mathbb{Z}$, and $\vec{e_i}$'s satisfy the relation (6). Then θ_{xy} for two embedding vectors $\vec{x}, \vec{y}$ is given by

$$\theta_{xy} = \sum_{i,j=1}^{n} (x_i \vec{e_i}) \cdot J_0 \ (y_j \vec{e_j}) = \sum_{i,j=1}^{n} x_i \theta_{ij} y_j. \tag{23}$$

Namely, we get

$$U_x U_y = \exp(2\pi i \, x^t \Theta y) U_y U_x, \tag{24}$$

where $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ belong to $\mathbb{Z}^n$. Under the action of $\rho(R)$, the transformed x is given by $x' = \rho(R)x = R^t x$, and thus

$$U_{x'} U_{y'} = \exp(2\pi i \, (R^t x)^t \Theta (R^t y)) U_{y'} U_{x'} = \exp(2\pi i \, x^t R\Theta R^t y) U_{y'} U_{x'}. \tag{25}$$

Therefore, the two tori with Θ and $\Theta' = R\Theta R^t$ are isomorphic.

For $\nu(N) \in SO(n, n, \mathbb{Z})$, $\Theta' = \nu(N)\Theta$ is given by replacing θ_{ij} with $\theta'_{ij} = \theta_{ij} + n_{ij}$ for $n_{ij} \in \mathbb{Z}$. This does not change the commutation relations among U_x, therefore $A^n_{\theta'}$ and A^n_{θ} correspond to the same noncommutative torus.

To prove that $A^n_{\theta'}$ with $\Theta' = \sigma_2 \Theta$ is Morita equivalent to A^n_{θ}, we first prove it for any σ_{2p} with a positive integer p obeying $2p \le n$. For this we consider $\Theta \in \mathcal{T}_n$ in 2×2 block form whose top left part is a $2p \times 2p$ matrix denoted by θ_{11},

$$\Theta := \begin{pmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{pmatrix}. \tag{26}$$

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Then

$$\Theta' = \sigma_{2p}\Theta = \begin{pmatrix} \theta_{11}^{-1} & -\theta_{11}^{-1}\theta_{12} \\ \theta_{21}\theta_{11}^{-1} & \theta_{22} - \theta_{21}\theta_{11}^{-1}\theta_{12} \end{pmatrix}, \tag{27}$$

and the noncommutative tori corresponding to Θ' and Θ are Morita equivalent.

To prove this we first choose an invertible matrix T_{11} such that $T_{11}^t J_0 T_{11} = -\theta_{11}$ where

$$J_0 = \begin{pmatrix} 0 & I_p \\ -I_p & 0 \end{pmatrix}.$$

Then set $T_{31} = \theta_{12}^t$, and T_{32} be any $q \times q$ matrix such that $\theta_{22} = T_{32}^t - T_{32}$ where $q := n - 2p$. Now, set a $(n+q) \times n$ matrix T as

$$T := \begin{pmatrix} T_{11} & 0 \\ 0 & I_q \\ T_{31} & T_{32} \end{pmatrix}, \tag{28}$$

and a $(n+q) \times (n+q)$ square matrix J as

$$J := \begin{pmatrix} J_0 & 0 & 0 \\ 0 & 0 & I_q \\ 0 & -I_q & 0 \end{pmatrix}, \tag{29}$$

then it can be shown that $T^t J T = -\Theta$. Note that a linear transformation T carries $\mathbb{Z}^{2p} \times \mathbb{Z}^q$ into $\mathbb{R}^{2p} \times \mathbb{Z}^q \times \mathbb{R}^q$. Thus T satisfies the conditions in [22] for being an embedding map. This means that we can view T as a homomorphism from $\mathbb{Z}^n$ into $\mathcal{G} = \mathbb{R}^{2p} \times \mathbb{Z}^q \times \mathbb{T}^q$, then the range of T is a lattice in $\mathcal{G}$, which we denoted by D. From the relation $\mathcal{G} = M \times \hat{M}$, here $M = \mathbb{R}^p \times \mathbb{Z}^q$. As we saw before, $\mathcal{G}$ carries the Heisenberg cocycle β defined by

$$\beta((m, \hat{s}), (n, \hat{t})) = \exp(2\pi i < m, \hat{t} >), \tag{30}$$

where $<,>$ denotes the pairing between M and $\hat{M}$. The corresponding skew cocycle ρ is defined by

$$\rho((m, \hat{s}), (n, \hat{t})) = \exp(2\pi i (< m, \hat{t} > - < n, \hat{s} >)). \tag{31}$$

Note that $\rho(x, y) = \exp(2\pi i \, x^t J y)$ for the J defined above. Since $T^t J T = -\Theta$, the restriction of ρ to D and so to $\mathbb{Z}^n$ is exactly $(x, y) \mapsto \exp(-2\pi i \, x^t \Theta y)$. Thus the algebra $A = \mathcal{S}(\mathbb{Z}^n, \beta^*)$, the space of Schwartz functions on $\mathbb{Z}^n$ with convolution twisted by the cocycle β^*, is isomorphic to the noncommutative torus A_θ^n and has a natural right action on the space $\mathcal{S}(M)$.

Now we like to know the endomorphism algebra of this right A-module, which is Morita equivalent to A via this module. In [22], this endomorphism algebra is given by $\mathcal{S}(D^\perp, \beta)$ where

$$D^\perp = \{w \in \mathcal{G} : \rho(w, z) = 1 \text{ for all } z \in D\}, \tag{32}$$

and β is restricted to $D^\perp$.

Our next step is to determine $D^\perp$ as the image of $\mathbb{Z}^n$ under some embedding map, such that from this embedding map we can get Θ' which on $\mathbb{Z}^n$ gives the cocycle corresponding to the restriction of ρ to $D^\perp$. Any given $x \in \mathcal{G}$ will be in $D^\perp$, exactly if $x^t J T z \in \mathbb{Z}$ for all $z \in \mathbb{Z}^n$, i.e., exactly if $T^t J x \in \mathbb{Z}^n$. For a natural isomorphism from $\mathbb{Z}^n$ to this $D^\perp$, we define an invertible $(n+q) \times (n+q)$ matrix

$$\bar{T} := \begin{pmatrix} T_{11} & 0 & 0 \\ 0 & I_q & 0 \\ T_{31} & T_{32} & I_q \end{pmatrix}. \tag{33}$$

It can be checked that $T^t J x \in \mathbb{Z}^n$ exactly if $\bar{T}^t J x \in \mathbb{Z}^{n+q}$. Since $\bar{T}, J$ are invertible, the following holds viewed in $\mathcal{G}$.

$$D^\perp = (\bar{T}^t J)^{-1}(\mathbb{Z}^{n+q}). \tag{34}$$

Since the inverse of $(\bar{T}^t J)$ is given by

$$(\bar{T}^t J)^{-1} = \begin{pmatrix} -J_0 (T_{11}^t)^{-1} & 0 & J_0 (T_{11}^t)^{-1} T_{31}^t \\ 0 & 0 & -I_q \\ 0 & I_q & -T_{32}^t \end{pmatrix}, \tag{35}$$

we get

$$(\bar{T}^t J)^{-1}(0 \times \mathbb{Z}^q \times 0) = 0 \times 0 \times \mathbb{Z}^q, \tag{36}$$

which is 0 in $\mathcal{G} = \mathbb{R}^{2p} \times \mathbb{Z}^q \times \mathbb{T}^q$. Thus omitting the second column of $(\bar{T}^t J)^{-1}$ and up to an inessential sign, we get the following desired embedding map

$$S = \begin{pmatrix} J_0 (T_{11}^t)^{-1} & -J_0 (T_{11}^t)^{-1} T_{31}^t \\ 0 & I_q \\ 0 & T_{32}^t \end{pmatrix}, \tag{37}$$

which gives an isomorphism from $\mathbb{Z}^n$ onto $D^\perp$. Now one can easily show that

$$S^t J S = \sigma_{2p} \Theta. \tag{38}$$

So far we have shown that the noncommutative torus corresponding to $\sigma_{2p} \Theta$ is Morita equivalent to that of Θ. However, in the above proof we have a freedom of decomposing the matrix Θ. Namely we have a freedom to choose the size of the component θ_{11} in (26), which is given by a $2p \times 2p$ matrix, for any p within the range of $2 \le 2p \le n$. This means that the noncommutative tori with $M = \mathbb{R}^p \times \mathbb{Z}^q$ of different p's can have the same Θ in this set-up. Thus the noncommutative tori with $\Theta' = \sigma_{2p} \Theta$ and $\Theta'' = \sigma_{2p'} \Theta$ are Morita equivalent. Therefore, the noncommutative tori corresponding to $\sigma_{2p} \Theta$ and $\sigma_2 \Theta$ are Morita equivalent. This proves the last part of the theorem.

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3 Noncommutative supertori

In this section we review the construction of noncommutative supertori in two dimensions [18] and comment on its possible extension to higher dimensions. In order to construct noncommutative supertori, we first consider the ambient noncommutative superspace. In the case of $\mathcal{N} = (1,1)$ ambient superspace, we found that the only deformation which preserves a part of supersymmetry is just giving the noncommutativity for bosonic coordinates whereas the fermionic coordinates are anticommuting each other [18]. So we will not consider this case. When we consider the case of $\mathcal{N} = (2,2)$ superspace spanned by $(X^1, X^2, \theta^\pm, \bar{\theta}^\pm)$[1], there are a few ways to give the non(anti)commutativity in this superspace. Here we consider the case of Q-deformation, where the Moyal product $*$ is given by the supercharge $Q_\pm$

$$* = \exp\left[\frac{i}{2}\Theta\epsilon^{\mu\nu}\frac{\overleftarrow{\partial}}{\partial X^\mu}\frac{\overrightarrow{\partial}}{\partial X^\nu} - \frac{C}{2}\left(\overleftarrow{Q_+}\overrightarrow{Q_-} + \overleftarrow{Q_-}\overrightarrow{Q_+}\right)\right]. \tag{39}$$

Here the supercharges $Q_\pm$, $\bar{Q}_\pm$ are defined by

$$Q_+ = \frac{\partial}{\partial\theta^+} - \frac{\bar{\theta}^+}{2}\left(\frac{\partial}{\partial X^1} - i\frac{\partial}{\partial X^2}\right) \qquad Q_- = \frac{\partial}{\partial\theta^-} - \frac{\bar{\theta}^-}{2}\left(\frac{\partial}{\partial X^1} + i\frac{\partial}{\partial X^2}\right)$$

$$\bar{Q}_+ = \frac{\partial}{\partial\bar{\theta}^+} - \frac{\theta^+}{2}\left(\frac{\partial}{\partial X^1} - i\frac{\partial}{\partial X^2}\right) \qquad \bar{Q}_- = \frac{\partial}{\partial\bar{\theta}^-} - \frac{\theta^-}{2}\left(\frac{\partial}{\partial X^1} + i\frac{\partial}{\partial X^2}\right). \tag{40}$$

We note that there is another non(anti)commutative deformation, called D-deformation. But the noncommutative supertorus with D-deformation does not have the structure of super Heisenberg group [18]. Now we move to the operator formalism. In Q-deformation, the nontrivial commutation relations among supercoordinates are

$$[X^1, X^2] = i\Theta - \frac{i}{2}C\bar{\theta}^+\bar{\theta}^-, \qquad [X^1, \theta^+] = \frac{1}{2}C\bar{\theta}^-, \qquad [X^1, \theta^-] = \frac{1}{2}C\bar{\theta}^+,$$

$$[X^2, \theta^+] = \frac{i}{2}C\bar{\theta}^-, \qquad [X^2, \theta^-] = -\frac{i}{2}C\bar{\theta}^+, \qquad [\theta^+, \theta^-] = C, \tag{41}$$

We note that the fermionic coordinates $\bar{\theta}^+, \bar{\theta}^-$ appear as the central elements. This algebra can be represented on the module $f(s, \eta, \bar{\theta}^+, \bar{\theta}^-)$ as

$$X^1 f(s, \eta, \bar{\theta}^+, \bar{\theta}^-) = \left(i\Theta\frac{\partial}{\partial s} + \frac{1}{2}\sqrt{C}\bar{\theta}^+\frac{\partial}{\partial\eta} + \frac{1}{2}\sqrt{C}\bar{\theta}^-\eta\right) f(s, \eta, \bar{\theta}^+, \bar{\theta}^-),$$

$$X^2 f(s, \eta, \bar{\theta}^+, \bar{\theta}^-) = \left(s - \frac{i}{2}\sqrt{C}\bar{\theta}^+\frac{\partial}{\partial\eta} + \frac{i}{2}\sqrt{C}\bar{\theta}^-\eta\right) f(s, \eta, \bar{\theta}^+, \bar{\theta}^-),$$

$$\theta^+ f(s, \eta, \bar{\theta}^+, \bar{\theta}^-) = \sqrt{C}\frac{\partial}{\partial\eta}f(s, \eta, \bar{\theta}^+, \bar{\theta}^-),$$

$$\theta^- f(s, \eta, \bar{\theta}^+, \bar{\theta}^-) = \sqrt{C}\eta f(s, \eta, \bar{\theta}^+, \bar{\theta}^-). \tag{42}$$

---

[1] the convention of complex conjugation is $(\theta^+)^* = \theta^-$, $(\bar{\theta}^+)^* = \bar{\theta}^-$.

Next we consider the $\mathcal{N} = (2,2)$ supertorus with the odd spin structure, In the case of the even spin structure, there is no effect from supersymmetrization [18]. Here, we set [2] the noncommutative parameters $\Theta = \frac{1}{2\pi}$ and $C = 1$. In this setup, the generators $U = U_1$ and $V = U_2$ satisfy

$$
\begin{aligned}
U X^\mu U^{-1} &= X^\mu + e_U^\mu, & U\theta^\pm U^{-1} &= \theta^\pm, & U\bar{\theta}^\pm U^{-1} &= \bar{\theta}^\pm, \\
V X^\mu V^{-1} &= X^\mu + e_V^\mu, & V\theta^\pm V^{-1} &= \theta^\pm + \delta^\pm, & V\bar{\theta}^\pm V^{-1} &= \bar{\theta}^\pm + \bar{\delta}^\pm,
\end{aligned}
\tag{43}
$$

where the supercoordinates obey the algebra (41) and the lattice vectors e_U^μ and e_V^μ are given by

$$
\begin{aligned}
e_U^\mu &= {}^t(1, 0), \\
e_V^\mu &= {}^t\big(\mathrm{Re}(\tau + \bar{\theta}^+ \delta^+ + \theta^+ \bar{\delta}^+),\ \mathrm{Im}(\tau + \bar{\theta}^+ \delta^+ + \theta^+ \bar{\delta}^+)\big).
\end{aligned}
\tag{44}
$$

Then the explicit form of U, V can be obtained as

$$
\begin{aligned}
U &= \exp(2\pi i s), \\
V &= \exp\left[2\pi i (\mathrm{Re}\,\tau) s + (\mathrm{Im}\,\tau)\frac{\partial}{\partial s} + \delta^+ \mathcal{Q}_+ + \delta^- \mathcal{Q}_-\right] \\
&= \exp\left[2\pi i (\mathrm{Re}\,\tau) s + (\mathrm{Im}\,\tau)\frac{\partial}{\partial s} + \delta^+ \eta + \delta^- \frac{\partial}{\partial \eta}\right],
\end{aligned}
\tag{45}
$$

where the operators $\mathcal{Q}_\pm$ are the representation of the supercharges $Q_\pm$ on the module, and can be simply represented by

$$
\mathcal{Q}_+ = \eta, \quad \mathcal{Q}_- = \frac{\partial}{\partial \eta}.
\tag{46}
$$

In the above we set $\bar{\delta}^\pm = 0$, since the representation of $\bar{Q}^\pm$ on the module is given by

$$
\bar{\mathcal{Q}}_+ = \frac{\partial}{\partial \bar{\theta}^+} - i\eta\left(2\pi s - \frac{\partial}{\partial s}\right), \quad \bar{\mathcal{Q}}_- = \frac{\partial}{\partial \bar{\theta}^-} - i\frac{\partial}{\partial \eta}\left(2\pi s - \frac{\partial}{\partial s}\right),
\tag{47}
$$

which contains the second order derivative[3], and thus it cannot be included in V due to the restriction of the structure of super Heisenberg group [18]. In general, when a commutator of any two generators corresponding to two of the basis vectors of the embedding map are nonvanshing non-constant, then the lattice translation vectors in (43) cannot be given by constants, thus the periodic property of torus cannot be obtained. This property of yielding constant commutators among generators is an essential requirement for Heisenberg group. The commutation relation between U and V is given by

$$
UV = \exp(-2\pi i\,\mathrm{Im}\,\tau) VU.
\tag{48}
$$

---

[2]This normalization is different from our previous work [18] by a factor of 2π.

[3]This corresponds to that the supersymmetry generated by $\bar{Q}^\pm$ is broken by the Q-deformation.

The embedding map can be written from (45) as

$$
\widetilde{\Phi}_Q =
\begin{array}{c}
\\
\frac{\partial}{\partial s} \\
s \\
\frac{\partial}{\partial \eta} \\
\eta
\end{array}
\begin{array}{c}
\;\; U \qquad V \\
\left(
\begin{array}{cc}
0 & \mathrm{Im}\,\tau \\
1 & \mathrm{Re}\,\tau \\
0 & \delta^- \\
0 & \delta^+
\end{array}
\right).
\end{array}
\tag{49}
$$

We can compare this with the bosonic case where the noncommutativity parameter θ_{ij} is given by (6). If we write $U := U_{\vec{E}_1}$ and $V := U_{\vec{E}_2}$ with supersymmetric basis vectors $\vec{E}_1$ and $\vec{E}_2$, then the relation (5) becomes

$$
U_{\vec{E}_1} U_{\vec{E}_2} = e^{2\pi i \widetilde{\theta}_{12}} U_{\vec{E}_2} U_{\vec{E}_1},
\tag{50}
$$

and the relation (6) becomes

$$
\widetilde{\theta}_{12} = \vec{E}_1 \cdot \widetilde{J}_0 \vec{E}_2 \quad \text{where} \quad
\widetilde{J}_0 =
\left(
\begin{array}{cc}
\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} & 0 \\
0 & \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}
\end{array}
\right).
\tag{51}
$$

Let us comment on the higher-dimensional case and the case with more extended supersymmetry. Since the latter case is obtained from the former case by dimensional reduction, we consider the former one. First of all, we like to recall that the supersymmetrization of noncommutative torus with the embedding map of $\mathbb{R}^p \times \mathbb{Z}^q$ type, where the dimension of torus is $n = 2p + q$ with nonzero q, is as yet unknown. This is because the supersymmetrization of noncommutative torus with nonzero q embedding case necessarily deals with a deformation of discrete(lattice) supersymmetry, and this is not well understood so far.[4]

---

[4]There have been some works on lattice supersymmetry in two different directions: One is to realize lattice supersymmetry by orbifolding of supersymmetric Yang-Mills theory [32] and the other is to construct consistent supersymmetry generators directly on lattice [33]. Although the first approach deals with the lattice formulation of supersymmetric Yang-Mills theory, here we need a noncommutive super-lattice formulation in the sense of Seiberg [12], not a superbundle on (bosonic) lattice, in order to construct the $\mathbb{R}^p \times \mathbb{Z}^q$ type embedding in the sense of Rieffel [22] for noncommutative supertori. Namely, in the embedding of noncommutative n-tori briefly mentioned in section 2, $M = \mathbb{R}^p \times \mathbb{Z}^q \times F$ where $2p + q = n$, the $\mathbb{Z}^q$-part really deals with the lattice nature of embedding manifold M while the "finite" F-part deals with the additional group structure of the bundle living on a torus. The first approach applies the orbifolding method to this F-part. This type of orbifolding the gauge bundles on noncommutative tori was also investigated earlier in [34]. In the second approach, so far no progress has been reported on noncommutative deformation. We thank M. Unsal for informing us the works done along the first approach.

Thus in this paper we only consider the vanishing q case, namely the supersymmetrization of noncommutative tori with $\mathbb{R}^p$ type embedding map.

In order to respect the property of Heisenberg group, we consider noncommutative supertori in higher dimensions with only odd spin structures and with Q-deformations. In this case, we can express the basis vectors in an embedding map as $\vec{E}_i = (x, \alpha) \in \mathbb{R}^{n|m}$ where $i = 1, \ldots, n$ and x, α are bosonic(Grassmann even) and fermionic(Grassmann odd) variables respectively. Notice also that in order to satisfy the structure of Heisenberg group we only consider the cases in which both n and m are even integers, say $n = 2p$ and $m = 2r$. Now the noncommutativity parameter can be expressed as

$$\widetilde{\theta}_{ij} = \vec{E}_i \cdot \widetilde{J}_0 \vec{E}_j \quad \text{where} \quad \widetilde{J}_0 = \begin{pmatrix} \begin{pmatrix} 0 & I_p \\ -I_p & 0 \end{pmatrix} & 0 \\ 0 & \begin{pmatrix} 0 & I_r \\ I_r & 0 \end{pmatrix} \end{pmatrix}. \tag{52}$$

Therefore the antisymmetric noncommutativity parameter matrix $\widetilde{\Theta}$ is given by

$$\widetilde{\Theta} = \widetilde{\Phi}^t \widetilde{J}_0 \widetilde{\Phi}, \tag{53}$$

as in the bosonic case (7). Note that the embedding map $\widetilde{\Phi}$ can be expressed in terms of basis vectors and can be decomposed of bosonic and fermionic parts as

$$\widetilde{\Phi} = \begin{pmatrix} \vec{E}_1 & \vec{E}_2 & \cdots & \vec{E}_n \end{pmatrix} := \begin{pmatrix} B \\ F \end{pmatrix}, \tag{54}$$

where B is the bosonic part of the embedding map given by a $2p \times 2p$ matrix (here $n = 2p$) and F is the fermionic part of the embedding map given by a $2r \times 2p$ matrix. Then (53) can be decomposed as

$$\widetilde{\Theta} = B^t J_0 B + F^t \hat{J}_0 F \quad \text{where} \quad J_0 = \begin{pmatrix} 0 & I_p \\ -I_p & 0 \end{pmatrix}, \quad \hat{J}_0 = \begin{pmatrix} 0 & I_r \\ I_r & 0 \end{pmatrix}. \tag{55}$$

In principle, in the $\mathbb{R}^p$ type embedding case there seems to be no obstruction to construct noncommutative supertori in higher dimensions. Namely, using the operators s, $\frac{\partial}{\partial s}$, etc. and (a part of) supercharges which satisfy super Heisenberg algebra, we can construct noncommutative supertori. In general, we should consider some additional coordinates on superspace such as harmonic superspace [30] or projective superspace [31], if we want to contain supersymmetric field theory in higher dimensions. Therefore even in the $\mathbb{R}^p$ type embedding case if we want to consider supersymmetric field theories on higher dimensional noncommutative supertori, we should extend our construction to include these additional coordinates. We leave this issue open as a future problem.

4 Morita equivalence in the supersymmetric case

In this section we consider Morita equivalence of noncommutative supertori in the $\mathbb{R}^p$ type embedding case. Let $\mathcal{V}$ be the Grassmann algebra over $\mathbb{C}$ generated by L elements, $\{e_1, ... e_L\}$, where L is large enough. Then $\mathcal{V} = V^0 \bigoplus V^1$, where V^0 consists of even elements and V^1 of odd elements. An element $x \in \mathcal{V} = \Sigma x_I E^I$, where I is a multi-index for the basis $\{E^I\}$, which consists of 2^L elements $\{1, e_i, e_i e_j, \cdots, \prod_{i=1}^{L} e_i\}$ and $x_I \in \mathbb{C}$. The body and soul of x can be defined as follows. The body of x denoted by x_0 contains no e_i, and the soul of x is $x - x_0$. Thus the body part belongs to V^0 and the soul part contains every element in $\mathcal{V}$ except the body part thereby containing at least one e_i. We can say that the noncommutativity parameters $\widetilde{\Theta}$ for noncommutative supertori in the previous section belongs to V^0 meaning that it can contain both body and soul.

Since we restrict ourselves to the $\mathbb{R}^p$ type embedding case in this paper, we cannot blindly follow the method used in section 2 for the bosonic case. Thus rather than assuming the symmetry group of Morita equivalence and proving it, we search for symmetry which yields Morita equivalence. We know from section 2 that the endomorphism algebra of the module of noncommutative torus is Morita equivalent to the given noncommutative torus and the condition is given by (32). This can be translated into

$$\Phi^t J_0 \Phi' = K, \tag{56}$$

where Φ is the embedding map of the given torus and Φ' is the embedding map of the dual torus, and K is an intersection matrix whose elements belong to $\mathbb{Z}$. Using the notation of the previous section, the above relation can be put into the following form in the supersymmetric case:

$$\widetilde{\Phi}^t \widetilde{J_0} \widetilde{\Phi}' = B^t J_0 B' + F^t \hat{J}_0 F' = \widetilde{K} \quad \text{where} \quad J_0 = \begin{pmatrix} 0 & I_p \\ -I_p & 0 \end{pmatrix}, \ \hat{J}_0 = \begin{pmatrix} 0 & I_r \\ I_r & 0 \end{pmatrix}. \tag{57}$$

Here $\widetilde{\Phi} := \begin{pmatrix} B \\ F \end{pmatrix}$ and $\widetilde{\Phi}' := \begin{pmatrix} B' \\ F' \end{pmatrix}$ are the embedding maps of the given supertorus and the dual supertorus, respectively, and the elements of the intersection matrix $\widetilde{K}$ belong to $\mathbb{Z}$. Note that the entries of B, B' and F, F' belong to V^0 and V^1, respectively. Below we will denote the antisymmetric noncommutativity parameter matrix $\widetilde{\Theta}$ as $\Theta_{B,F}$ to express that it contains both bosonic and fermionic contributions though $\widetilde{\Theta}$ itself belongs to V^0:

$$\widetilde{\Theta} := \Theta_{B,F} = \widetilde{\Phi}^t \widetilde{J_0} \widetilde{\Phi} = B^t J_0 B + F^t \hat{J}_0 F. \tag{58}$$

Note that under the change of basis, the intersection matrix $\widetilde{K}$ in the duality condition (57) can be any element in $GL(n, \mathbb{Z})$. And this induces a part of Morita equivalence symmetry as we see below.

From the condition (57) we can express the bosonic part of the dual embedding map as

$$B' = -J_0 B^{-t}(\widetilde{K} - F^t \hat{J}_0 F'). \tag{59}$$

Thus using the relation (58) we can express the noncommutativity matrix $\widetilde{\Theta}' := \Theta_{B',F'}$ of the dual supertorus as

$$
\begin{aligned}
\Theta_{B',F'} &= -(\widetilde{K} - F^t \hat{J}_0 F')^t B^{-1} J_0 J_0 J_0 B^{-t}(\widetilde{K} - F^t \hat{J}_0 F') + F'^t \hat{J}_0 F', \\
&= -(\widetilde{K} - F^t \hat{J}_0 F')^t (B^t J_0 B)^{-1}(\widetilde{K} - F^t \hat{J}_0 F') + F'^t \hat{J}_0 F', \tag{60}
\end{aligned}
$$

where we used $J_0^{-1} = -J_0$. We can also express $\Theta_{B',F'}$ directly using the dual embedding map $\widetilde{\Phi}'$. From the relation (57) we have

$$\widetilde{\Phi}' = (\widetilde{\Phi}^t \widetilde{J}_0)^{-1} \widetilde{K}, \tag{61}$$

thus

$$
\begin{aligned}
\Theta_{B',F'} &= \widetilde{\Phi}'^t \widetilde{J}_0 \widetilde{\Phi}' \\
&= \widetilde{K}^t (\widetilde{\Phi}^t \widetilde{J}_0)^{-t} \widetilde{J}_0 (\widetilde{\Phi}^t \widetilde{J}_0)^{-1} \widetilde{K} \\
&= \widetilde{K}^t (\widetilde{\Phi}^t \widetilde{J}_0^t \widetilde{\Phi})^{-1} \widetilde{K}. \tag{62}
\end{aligned}
$$

Since

$$\widetilde{J}_0^t = \begin{pmatrix} J_0 & 0 \\ 0 & \hat{J}_0 \end{pmatrix}^t = \begin{pmatrix} -J_0 & 0 \\ 0 & \hat{J}_0 \end{pmatrix},$$

we can write $\widetilde{\Phi}^t \widetilde{J}_0^t \widetilde{\Phi} = -\Theta_B + \Theta_F$ where we denote $\Theta_B := B^t J_0 B$ and $\Theta_F := F^t \hat{J}_0 F$. Therefore when the original bosonic part of the embedding map B consists of body only, then (62) can be written as

$$
\begin{aligned}
\Theta_{B',F'} &= \widetilde{K}^t (-\Theta_B + \Theta_F)^{-1} \widetilde{K} \\
&= -\widetilde{K}^t \Theta_B^{-1} \sum_{m=0}^{\infty} (\Theta_F \Theta_B^{-1})^m \widetilde{K}. \tag{63}
\end{aligned}
$$

Now, let us consider $\sigma_n = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}$ where n is the dimension of the torus. This belongs to the group $SO(n, n, V_{\mathbb{Z}}^0)$ defined by

$$SO(n, n, V_{\mathbb{Z}}^0) := \{g \in GL(2n, V_{\mathbb{Z}}^0) \mid g^t \hat{J}_{2n} g = \hat{J}_{2n}, \det g = 1\}, \quad \text{where} \quad \hat{J}_{2n} = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}$$

15

and $V_{\mathbb{Z}}^0$ denotes Grassmann even number(V^0) whose body part has the value in $\mathbb{Z}$. The group $SO(n, n, V_{\mathbb{Z}}^0)$ has the determinant 1, and yields the transformation which preserves the quadratic form $X_1 X_{n+1} + X_2 X_{n+2} + ... + X_n X_{2n}$. Thus the body part of the group must be in $SO(n, n, \mathbb{Z})$. The action of σ_n is given by (17) as in the bosonic case, thus

$$\sigma_n \Theta_{B,F} = \Theta_{B,F}^{-1} = (\Theta_B + \Theta_F)^{-1} = \Theta_B^{-1} \sum_{m=0}^{\infty} (-\Theta_F \Theta_B^{-1})^m. \tag{64}$$

One can see that this is just differ from $\Theta_{B',F'}$ in the soul part up to the actions of $\widetilde{K} \in GL(n, \mathbb{Z})$ which generates Morita equivalent tori as we see below. Furthermore, the relation (60) tells us that the soul part F' of the dual map is not restricted by the duality condition (57). Namely, if the two $\widetilde{\Theta}$'s have the same body parts and differed by the soul parts belonging to V^0, then the two corresponding tori are Morita equivalent. Thus we can say that σ_n generates Morita equivalent tori.

The above statement that the same body parts up to elements in V^0 yield equivalent tori dictates us another symmetry action of the following element of $SO(n, n, V_{\mathbb{Z}}^0)$

$$\nu(\widetilde{N}) = \begin{pmatrix} I_n & \widetilde{N} \\ 0 & I_n \end{pmatrix},$$

where $\widetilde{N}$ is an antisymmetric $n \times n$ matrix whose entries are in $V_{\mathbb{Z}}^0$. The action of $\nu(\widetilde{N})$ is given as before by (17)

$$\nu(\widetilde{N}) \Theta_{B,F} = \Theta_{B,F} + \widetilde{N}. \tag{65}$$

The "rotation" by $\rho(\tilde{R}) \in SO(n, n, V_{\mathbb{Z}}^0)$ can be similarly considered as in the bosonic case. We consider $\rho(\tilde{R}) \in SO(n, n, V_{\mathbb{Z}}^0)$ given by

$$\rho(\tilde{R}) = \begin{pmatrix} \tilde{R}^t & 0 \\ 0 & \tilde{R}^{-1} \end{pmatrix},$$

where $\tilde{R} \in GL(n, V_{\mathbb{Z}}^0)$. When a basis $\{\vec{E}_i\}$ ($i = 1, 2, \cdots, n$) is given, we can consider a general embedding vector $\vec{X}$ in terms of given basis such as $\vec{X} = \sum_1^n X_i \vec{E}_i$ where $X_i \in V_{\mathbb{Z}}^0$, and $\vec{E}_i$'s satisfy the relation (52). Then $\widetilde{\theta}_{XY}$ for two general embedding vectors $\vec{X}$ and $\vec{Y}$ we have

$$\widetilde{\theta}_{XY} = \sum_{i,j=1}^{n} (X_i \vec{E}_i) \cdot \widetilde{J}_0 \, (Y_j \vec{E}_j) = \sum_{i,j=1}^{n} X_i \widetilde{\theta}_{ij} Y_j. \tag{66}$$

Namely,

$$U_X U_Y = \exp(2\pi i \, X^t \widetilde{\Theta} Y) U_Y U_X, \tag{67}$$

16

where $X = (X_1, \ldots, X_n)$ and $Y = (Y_1, \ldots, Y_n)$. Under the action of $\rho(\tilde{R})$, the transformed X is given by $X' = \rho(\tilde{R})X = \tilde{R}^t X$, and thus

$$U_{X'} U_{Y'} = \exp(2\pi i \, (\tilde{R}^t X)^t \widetilde{\Theta}(\tilde{R}^t Y)) U_{Y'} U_{X'} = \exp(2\pi i \, X^t \tilde{R} \widetilde{\Theta} \tilde{R}^t Y) U_{Y'} U_{X'}. \qquad (68)$$

Therefore, the two tori with $\widetilde{\Theta}$ and $\widetilde{\Theta}' = \tilde{R} \widetilde{\Theta} \tilde{R}^t$ are isomorphic. In (60) we can see that the action of $\rho(\tilde{R})$ is already incorporated in the transformation to the dual torus. There we see that $\tilde{R}$ appears as $(\widetilde{K} - F^t \hat{J}_0 F')^t$ acting on the body part of $\sigma_n \widetilde{\Theta}$. Since $\widetilde{\Theta}$'s with the same body part are Morita equivalent, $\rho(\tilde{R}) \Theta_B{}^{-1}$ in (60) is equivalent to $\rho(\tilde{R}) \widetilde{\Theta}^{-1}$. Thus we may explain the transformation in (60) by the following diagram.

$$\begin{array}{ccc} \Theta_{B,F} & \rightarrow & \Theta_{B',F'} \\ \downarrow & & \uparrow \\ \Theta_{B,0} & \rightarrow & \Theta_{B',0} \end{array}$$

The left vertical map is obtained by $\nu(\widetilde{N} = -F^t \hat{J}_0 F)$, and the bottom horizontal map is just σ_n followed by the $\rho(\tilde{R} = \widetilde{K} - F^t \hat{J}_0 F')$. Then the right vertical map is $\nu(\widetilde{N} = F'^t \hat{J}_0 F')$. Then all the composition is the desired Morita equivalent map, the upper horizontal arrow.

So far, we have shown that the three elements $\rho(\tilde{R}), \nu(\widetilde{N})$ with $\tilde{R}, \widetilde{N} \in V_{\mathbb{Z}}^0$, and σ_n yield the Morita equivalent noncommutative n-supertori. One may also prove that the group $SO(n, n, V_{\mathbb{Z}}^0)$ is generated by $\rho(\tilde{R}), \nu(\widetilde{N})$ and σ_2 just as in the bosonic case. However, we could not show that σ_k with $k < n$ are symmetry elements yielding Morita equivalent tori due to the lack of $\mathbb{R}^p \times \mathbb{Z}^q$ type embedding construction with nonzero q as we stressed before. Therefore, although we cannot say that the group $SO(n, n, V_{\mathbb{Z}}^0)$ is the symmetry group of Morita equivalence of higher dimensional $(n > 2)$ noncommutative supertori, we have the following result in the two dimensional case.

Theorem: If $g \in SO(2, 2, V_{\mathbb{Z}}^0)$, where $V_{\mathbb{Z}}^0$ denotes Grassmann even number whose body part belongs to $\mathbb{Z}$, then the noncommutative supertorus corresponding to $g\widetilde{\Theta}$ is Morita equivalent to the noncommutative supertorus corresponding to $\widetilde{\Theta}$ in two dimensions.

5 Conclusion

In this paper, we show that the group $SO(2, 2, V_{\mathbb{Z}}^0)$ yields Morita equivalent noncommutative supertori in the two dimensional case. For the higher dimensional case $(n > 2)$, we almost obtain a similar result as in the bosonic case. Namely, we obtain the three elements belonging to the group $SO(n, n, V_{\mathbb{Z}}^0)$, which are $\rho(\tilde{R}), \nu(\widetilde{N})$ and σ_n that yield Morita equivalent

tori. However, due to the absence of $\mathbb{R}^p \times \mathbb{Z}^q$ type embedding construction with nonzero q in the supersymmetric case we do not have σ_k with $k < n$. Thus we are short of having $SO(n, n, V_{\mathbb{Z}}^0)$ as the symmetry group in the high dimensional case. However, this does not prove that σ_k with $k < n$ are not the symmetry elements giving Morita equivalent tori. At the present stage we simply do not know whether they are the symmetry elements or not due to our incomplete knowledge of higher dimensional noncommutative supertori. We thus leave the issue of understanding $\mathbb{R}^p \times \mathbb{Z}^q$ type embedding in the supersymmetric case as a future problem.

Acknowledgments

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